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The conventional form of the Cantor-Schröder-Bernstein Theorem in set theory is as follows:

Let A and B be sets, let f be a one-to-one function from A into B, and let g be a one-to-one function from B into A. Then there exists a one-to-one function h from A <u>onto</u> B.

This paper proves this theorem, and investigates five other mathematical settings to decide if analogous theorems can be formulated in these settings. Analogous theorems are shown to exist in vector spaces, in finite groups, in free abelian groups, and in divisible groups. Counterexamples are presented to demonstrate that no theorem analogous to the Cantor-Schröder-Bernstein Theorem can be formulated for topological spaces or for groups of infinite order.

ANALOGIES TO THE CANTOR-SCHRÖDER-BERNSTEIN THEOREM

by

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INTRODUCTION

Let A and B be sets, let f be a one-to-one function from A into B, and let g be a one-to-one function from B into A. The Cantor-Schröder-Bernstein Theorem states that there exists a one-to-one function h from A <u>onto</u> B. This theorem is well known, and has long been important in set theory. We present a proof of this theorem in Chapter I.

In mathematics, the concept of an <u>isomorphism</u> is of great importance. The word "isomorphism" means different things in different contexts. Perhaps one can intuitively define this term in general as follows: if X is a certain type of mathematical structure, consisting of a set and possibly one or more binary operations, relations, orders, or topologies; and if Y is a mathematical structure of the same type (meaning Y is a set with the same kinds of binary operations, relations, orders, or topologies); then X is isomorphic to Y provided there is a one-to-one function from X onto Y which preserves these binary operations, relations, orders, or topologies.

The above definition is more general than current usage of the word "isomorphism" warrants, but it does help explain the different uses of this word. For this reason some authors prefer to place an adjective before the word "isomorphism", thereby clarifying the context of the word for the reader. Hence one can find the terms "set isomorphism", "group isomorphism", "ring isomorphism", "order isomorphism", "vector space isomorphism", or "topological isomorphism" (more commonly, "homeomorphism") in the literature. This practice is not standard, however. For this reason, the adjective is sometimes placed in parenthesis, as in "(group) isomorphism", when it is used at all.

Consider the following specific example. If one is working in the context of set theory, where one's paramount concern is the size of the sets, then one defines two sets to be (set) isomorphic if there is simply a one-to-one function from one set onto the other (for in this context, there are no binary operations, relations, orders, or topologies being studied). Using this definition of (set) isomorphism, the Cantor-Schröder-Bernstein Theorem can be stated as follows:

(CSB) Let A and B be sets. If A is (set) isomorphic to a subset of B, and if B is (set) isomorphic to a subset of A, then A is (set) isomorphic to B.

If one studies this form of stating (CSB), then one might naturally ask the following question:

> Does this Theorem remain true if the adjective "set" is replaced by any of the other adjectives which are commonly used with the term "isomorphism", and if the word "subset" is replaced by the term for the appropriate sub-structure?

Specifically, is (CSB) still true if we substitute for "set" the words, "vector space", "topological space", "group", "free abelian group", or "divisible abelian group"? In this thesis, we present the answers to this question. We will refer to these substitutions in (CSB) as analogies to the Cantor-Schröder-Bernstein Theorem. An example of what we have in mind as an analogy is the following:

> Let A and B be vector spaces. If A is (vector space) isomorphic to a subspace of B, and if B is (vector space) isomorphic to a subspace of A, then A is (vector space) isomorphic to B.

This statement is in fact true, and we prove this in Chapter II, after we state some basic definitions and facts about vector spaces.

In Chapter III, we investigate the analogy in topological spaces. After reviewing some definitions, we present a counterexample which shows that the analogy for the Cantor-Schröder-Bernstein Theorem is false for topological spaces.

In Chapter IV, we discuss groups in general, and two types of groups; namely, free abelian groups, and divisible abelian groups. For the general case, we prove the analogy valid if both groups have finite order, but we present a counterexample to the analogy for groups of infinite order. In Section Two, we prove that the analogy is true for free abelian groups. Our proof depends on the fact that this type of group has a basis. The analogy is also true for divisible groups, and we prove this in Section Three.

We shall give the basic definitions and facts about each of the mathematical structures we consider in the section devoted to that concept, rather than collect all of those preliminary remarks together here. Some of our examples and proofs are based on very difficult theorems. Rather than reproduce a proof of these theorems here, we have given references in the literature to these results for the interested reader. To attempt to make this thesis completely self-contained would have nearly tripled its length, and would have obscured our real purpose.

The conclusion of a proof is denoted by the symbol □.

CHAPTER I

THE CANTOR-SCHRÖDER-BERNSTEIN THEOREM

The Cantor-Schröder-Bernstein Theorem may be proved using the following lemma. The proof presented here is similar to a proof by Cox [1].

<u>1.1 LEMMA</u>. Let A be a set, let B be a subset of A, and let f be a one-to-one function from A into B. Then there exists a one-to-one function h from A onto B.

<u>Proof</u>: If $A = B$, the identity function on A is such a function. If $B \subsetneq A$, define $C = \{y \in A \mid$ there exists x in $A - B$ such that $y = f^n(x)$ for some $n \geq 0\}$, where f^0 is the identity function, and, for each positive integer k, $f^k = f^{k-1} \circ f$. So $C = \bigcup_{n=0}^{\infty} \text{Image}\,[f^n(A - B)] = (A - B) \cup f(A - B) \cup \cdots \cup f^n(A - B) \cup \cdots$.

For each z in A, define $h(z)$ as follows: $h(z) = f(z)$ if z is in C, and $h(z) = z$ if z is not in C. To show that h is one-to-one, consider the following three cases.

(a) Let a and b be in C and $a \neq b$. Then $h(a) = f(a)$ and $h(b) = f(b)$. Since f is one-to-one, and since $a \neq b$, then $f(a) \neq f(b)$. Hence $h(a) \neq h(b)$.

(b) Let neither a nor b be in C, and $a \neq b$. Then $h(a) = a$ and $h(b) = b$, so that $h(a) \neq h(b)$.

(c) Let a be in C and b not be in C, and $a \neq b$. Then $h(a) = f(a)$ and $h(b) = b$. Suppose $h(a) = h(b)$, then $b = f(a)$.

Since a is in C, $a = f^n(x)$ for some $n \geq 0$ and x in $A - B$. Now $f(a) = f^{n+1}(x)$. So $b = f(a)$ implies that $b = f^{n+1}(x)$, which means that b is in C, a contradiction to our hypothesis. Hence $h(a) \neq h(b)$ for $a \neq b$, so h is one-to-one.

To show that h is onto, we must show that for each b in B, there exists a z in A such that $h(z) = b$. Thus choose b in B.

(a) If b is not in C, then $h(b) = b$.

(b) If b is in C, then $b = f^n(x)$ with $n \geq 1$, and x in $A - B$. Note that $n \geq 1$ because b is not in $A - B$. Now $b = f(f^{n-1}(x))$ with $f^{n-1}(x)$ in C. It follows that $h(f^{n-1}(x)) = f(f^{n-1}(x)) = b$.

We have shown that for every b in B there exists a z in A such that $h(z) = b$. Therefore, h is a one-to-one function from A onto B. □

<u>1.2 THE CANTOR-SCHRÖDER-BERNSTEIN THEOREM</u>. If A and B are sets, if f is a one-to-one function from A into B, and if g is a one-to-one function from B into A, then there exists a one-to-one function from A onto B.

<u>Proof</u>: Let A and B be sets. Let f be a one-to-one function from A into B, and let g be a one-to-one function from B into A. Then $g \circ f$ is a one-to-one function from A into $g(B)$. Since $g(B) \subseteq A$, there exists a one-to-one function h from A onto $g(B)$ by (1.1).

Now g is one-to-one and onto from B to $g(B)$, so that g^{-1} is one-to-one and onto from $g(B)$ to B. But h is one-to-one and onto from A to $g(B)$; hence $g^{-1} \circ h$ is a one-to-one and onto function from A to B. $\square$

CHAPTER II

VECTOR SPACES

In this setting, the analogy to the Cantor-Schröder-Bernstein Theorem is valid. We shall show this result (Theorem 2.8), after we present some elementary definitions and a needed proposition.

2.1 DEFINITION. Let F be a field and let V be an abelian additive group (see 4.1.1), such that there exists a scalar multiplication of V by F which associates with each $c \in F$ and each $\alpha \in V$ an element $c\,\alpha \in V$. Then V is a vector space over F provided, if 1 is the unity of F, that

(a) $1\,\alpha = \alpha$

(b) $(c_1\,c_2)\,\alpha = c_1(c_2\,\alpha)$

(c) $c(\alpha + \beta) = c\alpha + c\beta$

(d) $(c_1 + c_2)\,\alpha = c_1\,\alpha + c_2\,\alpha$

for all $c_1, c_2 \in F$ and $\alpha, \beta \in V$. We call the elements in F scalars, and we call the elements in V vectors.

2.2 DEFINITION. Let V be a vector space, and let $S \subseteq V$. We call S linearly independent provided if $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq S$, if $\{c_1, c_2, \cdots, c_n\} \subseteq F$, and if $\sum_{i=1}^{n} c_i\,\alpha_i = 0$, then $c_1 = c_2 = \cdots = c_n = 0$. We say that S spans or generates V provided if $\beta \in V$, then $\beta = \sum_{i=1}^{n} c_i\,\alpha_i$ for some $\{\alpha_1, \cdots, \alpha_n\} \subseteq S$ and $\{c_1, \cdots, c_n\} \subseteq F$.

We say S is a basis for V provided S is linearly independent and spans V.

It can be proved that if S is a basis for V, and if $\beta \in V$, then there exists a unique n-tuple of scalars $(c_1, c_2, \cdots, c_n)$ such that $\beta = \sum_{i=1}^{n} c_i \alpha_i$ for some unique basis vectors $\alpha_1, \cdots, \alpha_n$ in S. For example, see [3, p. 41].

It is also possible to prove that every vector space V over a field F has a basis, and furthermore that any two bases for V have the same cardinality; for example, see Lang [4, p. 86]. Hence, let us define the dimension of a vector space V, denoted $\dim V$, to be the cardinality of one of its bases.

2.3 DEFINITION. If V and W are two vector spaces over the same field F, then a linear transformation T from V to W is a function from V to W such that $T(c\alpha + \beta) = c\,T(\alpha) + T(\beta)$ for all vectors $\alpha, \beta \in V$ and scalars $c \in F$. We call T a (vector space) isomorphism provided T is a one-to-one and onto linear transformation.

2.4 PROPOSITION. Let V and W be vector spaces over a field F. Then V is (vector space) isomorphic to W if and only if there exists a basis S for V and a basis R for W with the same cardinality, so $|S| = |R|$. That is, V is (vector space) isomorphic to W if and only if $\dim V = \dim W$.

Proof: $(\leftarrow)$ Let V and W be vector spaces over a field F. Suppose there exists a basis S for V and a basis R for W with $|S| = |R|$. Then there exists a function T which is one-to-one and onto from S to R. We can denote $S = \{\alpha_i\}_{i\in I}$ and $R = \{\beta_i\}_{i\in I}$

for some index set I, and assume $T : \alpha_i \to \beta_i$.

For any vector σ in V, there exists a unique k-tuple of scalars $(x_1, x_2, \cdots, x_k)$ such that $\sigma = x_1 \alpha_{i1} + x_2 \alpha_{i2} + \cdots + x_k \alpha_{ik}$ for some unique basis vectors $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ik}$ in S, by the definition of a basis.

Define T from V to W by
$T(\sigma) = x_1 \beta_{i1} + x_2 \beta_{i2} + \cdots + x_k \beta_{ik}$, where σ in V is defined as above. Thus we have extended T to a function from V to W. We will now proceed to show that T is an isomorphism from V onto W.

(a) To establish that T is one-to-one, let γ and θ be elements of V and let $T(\gamma) = T(\theta)$. We must show that $\gamma = \theta$.

Let $\gamma = x_1 \alpha_{i1} + \cdots + x_k \alpha_{ik}$ and $\theta = y_1 \alpha_{j1} + \cdots + y_m \alpha_{jm}$, where we assume that $x_i \neq 0$ and $y_j \neq 0$ for all i, j. Now,
$T(\gamma) = x_1 \beta_{i1} + x_2 \beta_{i2} + \cdots + x_k \beta_{ik}$ and
$T(\theta) = y_1 \beta_{j1} + y_2 \beta_{j2} + \cdots + y_m \beta_{jm}$. But, $T(\gamma) = T(\theta)$, so
$x_1 \beta_{i1} + x_2 \beta_{i2} + \cdots + x_k \beta_{ik} = y_1 \beta_{j1} + y_2 \beta_{j2} + \cdots + y_m \beta_{jm}$. Now, $\beta_{i1} \cdots \beta_{ik}$ and $\beta_{j1} \cdots \beta_{jm}$ are elements of R, and each vector in W (including $T(\gamma)$ and $T(\theta)$) can be expressed uniquely as a linear combination of elements of R. Thus we may assume, after reindexing, if necessary, that $k = m$ and that $x_1 = y_1, x_2 = y_2, \cdots, x_k = y_k$, and further that $\beta_{i1} = \beta_{j1}, \beta_{i2} = \beta_{j2}, \cdots, \beta_{ik} = \beta_{jk}$. So $\theta = \gamma$ and T is one-to-one.

(b) Now we show that T is onto. If λ is any vector in W, then there exists a unique k-tuple of scalars $(x_1, \cdots, x_k)$ and unique basis vectors $\beta_{i1}, \cdots, \beta_{ik}$ in R such that

$\lambda = x_1 \beta_{i1} + \cdots + x_k \beta_{ik}$. By our definition of T, $\lambda = T(x_1 \alpha_{i1} + \cdots + x_k \alpha_{ik})$. Clearly λ is in the range of T, proving that T is onto.

(c) T is also linear, by the following argument. Let θ and γ be members of V. Then we may write $\theta = x_1 \alpha_{i1} + \cdots + x_k \alpha_{ik}$ for some scalars $(x_1, \cdots, x_k)$ in F, and $\gamma = y_1 \alpha_{i1} + \cdots + y_k \alpha_{ik}$ for some scalars $(y_1, \cdots, y_k)$ in F, where it may be that $x_i = 0$ or $y_j = 0$ for some i, j. Let c be a scalar in F. Then

$$c\theta + \gamma = (cx_1 + y_1) \alpha_{i1} + \cdots + (cx_k + y_k) \alpha_{ik}, \text{ and}$$

$$T(c\theta+\gamma) = (cx_1 + y_1) \beta_{i1} + \cdots + (cx_k + y_k) \beta_{ik} =$$

$$\sum_{j=1}^{k} (cx_j + y_j) \beta_{ij} = c \sum_{j=1}^{k} x_j \beta_{ij} + \sum_{j=1}^{k} y_i \beta_{ij} =$$

$$c(T(\theta)) + T(\gamma).$$

Therefore, T is linear, and T is an isomorphism from V onto W.

($\rightarrow$) Suppose T is an isomorphism from V onto W. Also let $S = \{\alpha_i\}_{i\in I}$ be a basis for V. We will show that $R = \{T(\alpha_i)\}_{i\in I}$ is a basis for W. Notice that $|S| = |R|$.

Let $\beta \in W$. Since T is onto, there exists $\alpha \in V$ such that $\beta = T(\alpha)$. But α is in V and S is a basis for V, hence there exists a unique finite subset $\{\alpha_1, \cdots, \alpha_k\}$ of S and unique scalars $(x_1, \cdots, x_k)$ of F such that $\alpha = x_1 \alpha_1 + x_2 \alpha_2 + \cdots + x_k \alpha_k$. Now $\beta = T(\alpha)$ is in W, and $T(\alpha) = T(x_1 \alpha_1) + T(x_2 \alpha_2) + \cdots + T(x_k \alpha_k)$, so $\beta = T(\alpha) = x_1 T(\alpha_1) + x_2 T(\alpha_2) + \cdots + x_k T(\alpha_k)$. Therefore $\{T(\alpha_i)\}_{i\in I} = R$ generates W.

We will now show that R is linearly independent also, thereby qualifying as a basis for W.

Let c_j be scalars in F, and let $\sum_{j=1}^{k} c_j\ (T(\alpha_{ij})) = 0$ for some finite subset $\{T(\alpha_{ij}) \mid j = 1, \cdots, k\}$ of R. Then $T(\sum_{j=1}^{k} c_j\ \alpha_{ij}) = 0$, because T is linear. Hence $\sum_{j=1}^{k} c_j\ \alpha_{ij} = 0$, since T is one-to-one. Now S is linearly independent, thus $c_1 = \cdots = c_k = 0$. So we can generate the zero vector with a finite number of vectors from R only when all scalars equal 0; therefore R is linearly independent.

Thus, R is a basis for W, and dim V = dim W. □

A few well-known lemmas will now be discussed, and then these will be used in our next proof. For notational convenience in what follows, let us denote two vector spaces over a field F by V and W, and we will let T_1 be a one-to-one linear transformation from V into W and let T_2 be a one-to-one linear transformation from W into V. The range of T_1, denoted $rg(T_1)$, is $\{T_1(\alpha) \mid \alpha \text{ in } V\}$; similarly $rg(T_2) = \{T_2\ (\beta) \mid \beta \text{ in } W\}$. The kernel of T_1, denoted $\ker(T_1)$, is all α in V such that $T_1(\alpha) = 0$, and $\ker\ (T_2) = \{\beta \in W \mid T_2(\beta) = 0\}$. It is well-known that $rg(T_i)$ and $\ker\ (T_i)$ are subspaces of V and W, respectively, for $i = 1, 2$.

<u>2.5 LEMMA</u>. $\dim rg(T_1) \leq \dim W$ and $\dim rg(T_2) \leq \dim V$.

For proofs of 2.5 and 2.6, see Lang [4, p. 87 and p. 88].

<u>2.6 LEMMA</u>. $\dim \ker(T_1) + \dim rg(T_1) = \dim V$ and $\dim \ker(T_2) + \dim rg(T_2) = \dim W$.

Note that $\dim \ker(T_1) = 0$ and $\dim \ker(T_2) = 0$, since both T_1 and T_2 are one-to-one; so we have by 2.6 the following lemma.

<u>2.7 LEMMA</u>. $\dim rg(T_1) = \dim V$ and $\dim rg(T_2) = \dim W$.

Thus by 2.5 and 2.7, we have that $\dim V \leq \dim W$ and $\dim W \leq \dim V$; hence, $\dim V = \dim W$.

We are now equipped to state and prove a theorem for vector spaces analogous to the Cantor-Schröder-Bernstein Theorem.

2.8 THEOREM. If V and W are vector spaces over a field F, if V is (vector space) isomorphic to a subspace of W, and if W is (vector space) isomorphic to a subspace of V, then V is (vector space) isomorphic to W.

Proof: Let V and W be vector spaces over a field F. Let T_1 be a one-to-one linear transformation from V to W. Let T_2 be a one-to-one linear transformation from W into V. Then V is isomorphic to $rg(T_1)$ and W is isomorphic to $rg(T_2)$.

Applying 2.5, 2.6, 2.7, we have that $\dim V = \dim W$, and we conclude, by 2.4, that V is isomorphic to W. $\square$

CHAPTER III

TOPOLOGICAL SPACES

The analogy to the Cantor-Schröder-Bernstein Theorem in this setting is false, and we next develop the machinery necessary to exhibit a counterexample. First we shall review some elementary definitions.

3.1 DEFINITION. A topology on a nonempty set X is a collection $\mathcal{T}$ of subsets of X, called the $\mathcal{T}$-open sets, satisfying:

(a) X and ϕ are members of $\mathcal{T}$;

(b) the union of any number of elements of $\mathcal{T}$ belongs to $\mathcal{T}$;

(c) the intersection of any two elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

We say $(X, \mathcal{T})$ is a topological space if X is a set and if $\mathcal{T}$ is a topology on X.

3.2 DEFINITION. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. A function f from $(X, \mathcal{T})$ to $(Y, \mathcal{S})$ is continuous if and only if for each $\mathcal{S}$-Open set G in Y, $f^{-1}(G)$ is a $\mathcal{T}$-open set in X.

3.3 DEFINITION. Two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are homeomorphic if and only if there exists a one-to-one continuous function f from $(X, \mathcal{T})$ onto $(Y, \mathcal{S})$ such that f^{-1} is also continuous.

3.4 DEFINITION. We call a property of topological spaces a topological property if, whenever one space has the property, then so does every space which is homeomorphic to it.

<u>3.5 DEFINITION</u>. A <u>cover</u> of a space X is a collection $\mathcal{A}$ of subsets of X whose union is all of X. A <u>finite subcover</u> of a cover $\mathcal{A}$ is a finite subcollection $\mathcal{A}'$ of $\mathcal{A}$ which is also a cover. If the cover consists of open sets, it is referred to as an <u>open cover</u>.

<u>3.6 DEFINITION</u>. A space S is <u>compact</u> if and only if each open cover of X has a finite subcover.

The analogy to the Cantor-Schröder-Bernstein Theorem for topological spaces would be as follows:

3.7 <u>Let</u> $(X, \mathcal{T})$ <u>and</u> $(Y, \mathcal{S})$ <u>be two topological spaces</u>, <u>If</u> f <u>is a homeomorphism from</u> $(X, \mathcal{T})$ <u>onto a subspace of</u> $(Y, \mathcal{S})$, <u>and if</u> g <u>is a homeomorphism from</u> $(Y, \mathcal{S})$ <u>onto a subspace of</u> $(X, \mathcal{T})$, <u>then there exists a homeomorphism</u> h <u>from</u> $(X, \mathcal{T})$ <u>onto</u> $(Y, \mathcal{S})$.

However, as indicated earlier, 3.7 is not true, and we next present a counterexample.

<u>3.8 COUNTEREXAMPLE</u>. Let $(X, \mathcal{T})$ be the set of real numbers, $\mathcal{R}$, with the usual topology $\mathcal{T}$; i.e., open sets in $\mathcal{T}$ are open intervals $(a, b) = \{x \in \mathcal{R} \mid a < x < b\}$, where a and b are real numbers and $a < b$. Let $(Y, \mathcal{S})$ be $[0,1]$ with the usual subspace topology $\mathcal{S}$ of open sets $(a, b) \cap [0,1]$, $a < b$, where $a, b \in \mathcal{R}$.

Define f from $(Y, \mathcal{S})$ to $(X, \mathcal{T})$ by $f(x) = x$. Then f is a homeomorphism from $[0,1]$ onto a subspace of $\mathcal{R}$; that is, $(Y, \mathcal{S})$ onto a subspace of $(X, \mathcal{T})$.

To get a homeomorphism from $(X, \mathcal{T})$ onto a subspace of $(Y, \mathcal{S})$, we will need to define two functions. First, let α be defined by $\alpha(x) = \text{Arc tan } x$, for $x \in \mathcal{R}$. Now α is a homeomorphism from $\mathcal{R}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Define β by $\beta(x) = \frac{x}{\pi} + \frac{1}{2}$, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then β is a homeomorphism from $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $(0,1)$. By the transitive property of homeomorphisms, $\mathcal{R}$ is homeomorphic to $(0,1)$ under $\beta \circ \alpha$. So $\beta \circ \alpha$ is a homeomorphism from $\mathcal{R}$ onto a subspace of $[0,1]$.

We have established homeomorphisms from $(X, \mathcal{T})$ onto a subspace of $(Y, \mathcal{S})$ and from $(Y, \mathcal{S})$ onto a subspace of $(X, \mathcal{T})$. We need to show that there is no homeomorphism from $(X, \mathcal{T})$ onto $(Y, \mathcal{S})$.

The unit interval $[0,1]$ is a closed, bounded subset of $\mathcal{R}$, therefore $[0,1]$ is compact. But $\mathcal{R}$ is not compact, since $\bigcup_{n=1}^{\infty} (-n,n)$ is an open cover of $\mathcal{R}$ which clearly has no finite subcover. For a discussion of the above see Willard [5, p. 116 and p. 120].

It is well-known that compactness is a topological property. Therefore $\mathcal{R}$ is not homeomorphic to $[0,1]$, and the counterexample is completed.

CHAPTER IV

GROUPS

In this chapter we will show that the analogy to the Cantor-Schröder-Bernstein Theorem is invalid for groups in general, but is valid for certain types of groups; namely finite groups, free abelian groups and divisible groups.

SECTION 1: The General Case

We will prove that the analogy is valid when both groups are of finite order, but a counterexample will show that the analogy is invalid if the groups have infinite order. We begin by reviewing some well-known, but pertinent, definitions and lemmas.

4.1.1 DEFINITION. A group $(G, \oplus)$ is a non-empty set G of elements, together with a binary operation $\oplus$ on G such that:

i) $\oplus$ is associative;

ii) there exists an identity element $u \in G$ such that $u \oplus g = g = g \oplus u$ for all $g \in G$;

iii) if $a \in G$, then there exists an element a^{-1} in G with $a \oplus a^{-1} = u = a^{-1} \oplus a$. We call a^{-1} the inverse of a.

It is easy to prove that the identity element u in G is unique, and that the inverse of a is unique for every $a \in G$.

If $\oplus$ is commutative in the group $(G, \oplus)$, then we call $(G, \oplus)$ an abelian group.

<u>4.1.2 DEFINITION</u>. The <u>order</u> of a group $(G, \oplus)$, denoted $|G|$, is the number of elements in the set G; that is, the cardinal number of G.

<u>4.1.3 DEFINITION</u>. A <u>homomorphism</u> from a group $(G, \oplus)$ to a group $(H, *)$ is a function α which preserves the binary operations; that is, $\alpha(g_1 \oplus g_2) = \alpha(g_1) * \alpha(g_2)$ for all $g_1, g_2 \in G$.

It quickly becomes tedious to adopt a different symbol for each binary operation in each group; thus, after this point we shall use addition $+$ as the binary operation in each group. This convention agrees with current practice in the literature on abelian groups, and we shall soon be dealing primarily with abelian groups. Thus it is no longer necessary to refer to the group $(G, +)$; we shall simply refer to the group G, and let the binary operation $+$ be understood.

<u>4.1.4 DEFINITION</u>. An <u>isomorphism</u> from a group G to a group H is a one-to-one homomorphism from G onto H. We write $G \cong H$ if such an isomorphism exists, and we say that G is <u>isomorphic</u> to H.

We are now ready to state and prove the analogy to the Cantor-Schröder-Bernstein Theorem for finite groups.

<u>4.1.5 THEOREM</u>. Let G and H be two finite groups. If G is (group) isomorphic to a subgroup of H, and if H is (group) isomorphic to a subgroup of G, then G is (group) isomorphic to H.

<u>Proof</u>: Let G and H be two finite groups. Let G be isomorphic to a subgroup of H. Then there exists a one-to-one homomorphism α from G into H; hence $|G| \le |H|$. Let H be isomorphic to a subgroup of G; hence $|H| \le |G|$.

Now, $|G| \leq |H|$ and $|H| \leq |G|$; thus $|H| = |G|$. It is well-known that if two finite sets, G and H, have the same number of elements, and if there exists a one-to-one function α from G into H, then α is also onto. Thus α is an isomorphism of G onto H. $\square$

So the analogy is valid when both groups are finite. Now we need not consider the case where G is a finite group and H is an infinite group, because in this case H could not be isomorphic to a subgroup of G. Thus the hypothesis to the analogy could not be satisfied.

The final case to investigate involves two infinite groups. From this point, until the end of the section, we shall be considering only abelian groups. We will need some more definitions and another theorem.

4.1.6 DEFINITION. Let Z denote the integers. A group G is cyclic if all its elements are powers of some one element; that is, if $G = \{n\,g \mid n \in Z\}$ for some $g \in G$. In this case, we say that g generates G.

4.1.7 DEFINITION. If $a \in G$, and if there exists a positive integer n such that $na = 0$, where 0 denotes the identity element of G, then the smallest such positive integer is called the order of a. If no such positive integer exists, then a is said to have infinite order.

If g generates G and also has finite order n, then it can be proved that G is cyclic of order n and is isomorphic to the additive group of integers mod n, denoted Z_n.

If g generates G and also has infinite order, then it can be proved that G is isomorphic to the additive group of integers, denoted $\mathbb{Z}$. For a proof of the above two statements, consult Fuchs [2, p. 14].

<u>4.1.8 DEFINITION</u>. Let p be a prime number. A <u>p-primary</u> group, or just <u>p-group</u>, is a group all of whose elements have order some power of the fixed prime p.

<u>4.1.9 DEFINITION</u>. Let G and $G_1, G_2, \cdots, G_n, \cdots$ be abelian groups. We make the cartesian product $G_1 \times G_2 \times \cdots \times G_n \times \cdots$ into an abelian group by defining a binary operation $+$ as follows: if $x = (x_1, x_2, \cdots, x_n, \cdots)$ and $y = (y_1, y_2, \cdots, y_n, \cdots)$ are any two elements in this cartesian product, then $x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n, \cdots)$. We call this group the <u>direct sum</u> of $G_1, G_2, \cdots, G_n, \cdots$, and we denote it by $G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus \cdots$, or by $\oplus \sum_{i=1}^{\infty} G_i$. If I is any index set, and if $\{G_i \mid i \in I\}$ is a set of abelian groups, then we define $\oplus \sum_{i \in I} G_i$ in a similar manner.

<u>4.1.10 DEFINITION</u>. If $G \cong \oplus \sum_{i \in I} G_i$, we call this a <u>decomposition</u> of G into direct sums. Two decompositions of G into direct sums, $G \cong \oplus \sum_{i \in I} B_i$ and $G \cong \oplus \sum_{i \in I} C_i$, are called <u>isomorphic decompositions</u> if there exists a one-to-one correspondence between the two sets of components B_i and C_j such that corresponding components are isomorphic groups.

<u>4.1.11 DEFINITION</u>. If $G = \oplus \sum_{i=1}^{\infty} G_i$; then for each positive integer i, $\pi_i : G \to G_i$ defined by $\pi_i((x_1, x_2 \cdots, x_i, \cdots)) = x_i$,

where $x_j \in G_j$, is an onto homomorphism, and is called the natural projection homomorphism.

4.1.12 THEOREM. Any two decompositions of a group into direct sums of cyclic groups of prime power orders are isomorphic. For a proof of this see Fuchs, [2, p. 89].

A counterexample will now be exhibited to show that the analogy to the Cantor-Schröder-Bernstein Theorem is invalid for infinite groups.

4.1.13 COUNTEREXAMPLE. Consider the following groups:

$$G = Z_2 \oplus Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots,$$

and $$H = Z_4 \oplus Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots.$$

Notice that each decomposition is a direct sum of cyclic groups of prime power order, where the prime number is 2.

Define $\alpha : G \to H$ as follows:

If $y = (y_1, y_2, \cdots, y_n, \cdots) \in G$ (where all but a finite number of $y_i = 0$) then

$$\pi_1\ (\alpha(y)) = \begin{cases} 0 & \text{if } y_1 = 0 \\ 2 & \text{if } y_1 = 1 \end{cases} \qquad \text{and}$$

$$\pi_i\ (\alpha(y)) = y_i \quad \text{for all } i \geq 2.$$

A diagram of α is as follows: define $f : Z_2 \to Z_4$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \end{cases}.$$

Then f is a one-to-one homomorphism, and α may be pictured as:

$$G = Z_2 \oplus Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots$$

$$f\downarrow \quad id\downarrow \quad id\downarrow \quad id\downarrow$$

$$H = Z_4 \oplus Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots$$

We will now show that α is an isomorphism from G into H. Let $g_1 = (y_1, y_2, \cdots, y_n, \cdots) \in G$ and let $g_2 = (x_1, x_2, \cdots, x_n, \cdots) \in G$. We have four cases to consider: x_1 and $y_1 = 1$, x_1 and $y_1 = 0$, $x_1 = 1$ and $y_1 = 0$, or $x_1 = 0$ and $y_1 = 1$. We will prove this only for the case $x_1 = 1$ and $y_1 = 0$, since the other cases may be proved similarly. Now,

$\alpha(g_1 \oplus g_2) = \alpha(y_1 + x_1, \cdots, y_n + x_n, \cdots) = (2, \cdots, y_n + x_n, \cdots)$ and

$\alpha(g_1) \oplus \alpha(g_2) = (0, y_2, \cdots, y_n, \cdots) + (2, x_2, \cdots, x_n, \cdots) =$

$(2, \cdots, y_n + x_n, \cdots)$, so $\alpha(g_1 \oplus g_2) = \alpha(g_1) \oplus \alpha(g_2)$.

Thus α preserves the binary operation $\oplus$. Since α is clearly one-to-one, α is an isomorphism from G into H.

Define $\beta : H \to G$ as follows:

If $x = (x_1, x_2, \cdots, x_n, \cdots) \in H$ (and all but a finite number of $x_i = 0$), let $\beta(x) = (0, x_1, x_2, \cdots, x_n, \cdots)$; that is $\pi_1 (\beta(x)) = 0$ and $\pi_{i+1}\, \beta(x) = x_i$, where π_{i+1} is the natural projection map of $G \to G_{i+1}$, for all $i \geq 1$. We will show that β is an isomorphism from H into G. Let $h_1 = (y_1, y_2, \cdots, y_n, \cdots) \in H$ and $h_2 = (x_1, x_2, \cdots, x_n, \cdots) \in H$. Now,

$\beta(h_1 \oplus h_2) = \beta(y_1 + x_1, \cdots, y_n + x_n, \cdots) =$

$(0, y_1 + x_1, \cdots, y_n + x_n, \cdots)$ and $\beta(h_1) \oplus \beta(h_2) =$

$(0, y_1, \cdots, y_n, \cdots) + (0, x_1, \cdots, x_n, \cdots) =$

$(0, y_1 + x_1, \cdots, y_n + x_n, \cdots)$, so $\beta(h_1 \oplus h_2) = \beta(h_1) \oplus \beta(h_2)$.

Now, β is obviously one-to-one, and therefore is an isomorphism from H into G. A diagram of β is as follows:

$$\begin{array}{l} G = Z_2 \oplus Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots \\ \quad\quad\ \ \mathrm{id}\nearrow \quad \mathrm{id}\nearrow \quad \mathrm{id}\nearrow \\ H = Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots \quad . \end{array}$$

We now have G isomorphic to a subgroup of H and H isomorphic to a subgroup of G. We need to show that G is not isomorphic to H.

If $G \cong H$, their decompositions into direct sums of cyclic groups of order a power of 2 would be isomorphic, by 4.1.12. If $Z_2 \oplus Z_4 \oplus Z_4 \oplus \cdots \cong Z_4 \oplus Z_4 \oplus Z_4 \oplus \cdots$ we would be able to find a one-to-one correspondence between corresponding components of the decompositions so that $Z_2 \cong Z_4$, $Z_4 \cong Z_4$, $\cdots$. But clearly Z_2 is not isomorphic to Z_4.

Since the decompositions of G and H into direct sums of cyclic groups of order a power of 2 are not isomorphic, then, by the contrapositive of 4.1.12, G and H are not isomorphic groups. Thus there is no analogy to the Cantor-Schröder-Bernstein Theorem for infinite groups.

SECTION 2: <u>Free Abelian Groups</u>

<u>4.2.1 DEFINITION</u>. Let S be a set; the <u>free abelian group</u> F_S <u>on S</u> is $\oplus\,\Sigma_S\, Z$, where $\oplus\,\Sigma_S\, Z$ means the direct sum of $|S|$ copies of the integers Z. We say F is a <u>free abelian group</u> if there exists a set S such that $F \cong F_S$.

Let $i \in S$, and let α_i denote the element of F_S which is 1 in the i^{th} coordinate and 0 elsewhere. The set $\{\alpha_i \mid i \in S\}$ is a basis for $\underline{F_S}$, since every $x = (x_1, x_2, \cdots, x_k, \cdots) \in F_S$ can be uniquely written as a finite sum $x = \Sigma_S x_i \alpha_i$ (recall that all but a finite number of the components x_i are 0).

4.2.2 EXAMPLE. Let $S = \{1, 2, 3\}$. Then $F_S = Z \oplus Z \oplus Z = \{(n_1, n_2, n_3) \mid n_i \in Z\}$. We define $(n_1, n_2, n_3) + (m_1, m_2, m_3) = (n_1 + m_1, n_2 + m_2, n_3 + m_3)$, and for $k \in Z$, $k(n_1, n_2, n_3) = (kn_1, kn_2, kn_3)$. The set $\{\alpha_i \mid i \in S\}$ will consist of the following: $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$.

Choose $x = (n_1, n_2, n_3) \in F_S$; then

$x = (n_1, 0, 0) + (0, n_2, 0) + (0, 0, n_3) =$

$n_1(1, 0, 0) + n_2(0, 1, 0) + n_3(0, 0, 1) =$

$n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3$, therefore $\{\alpha_i \mid i \in S\}$ generates F_S.

Now, if $(0, 0, 0) = k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3$, then $(0, 0, 0) = k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (k_1, k_2, k_3)$; so $k_1 = 0$, $k_2 = 0$, $k_3 = 0$, hence $\{\alpha_i \mid i \in S\}$ is also linearly independent (the definitions of linearly independent and generate are the same as for vector spaces, except that the scalars are integers. See Definition 2.2).

Three important theorems will now be stated which will be needed later in the proof of the analogy for this section.

4.2.3 THEOREM. An abelian group G has a basis if and only if G is a free abelian group.

Proof: ($\leftarrow$) If G is a free abelian group, then there exists a set S such that $G \cong F_S$, and the set $\{\alpha_i \mid i \in S\}$ as defined above is a basis for F_S. Thus if $f : F_S \to G$ is an isomorphism, then $\{f(\alpha_i) \mid i \in S\}$ is a basis for G.

($\rightarrow$) Let G be an abelian group with a basis $\{x_\alpha\}_{\alpha\in S}$. Then every element $g \in G$ can be written uniquely as $g = \sum_{i=1}^{k} n_{\alpha i}\, x_{\alpha i}$, where $x_{\alpha i}$ is an element of $\{x_\alpha\}_{\alpha\in S}$ and where $n_{\alpha i} \in \mathbb{Z}$. So $G \cong F_S$, where $g \in G$ is mapped to the element in F_S which has as its α^{th} coordinate n_α.

4.2.4 THEOREM. Let A be a free abelian group, and let B be a subgroup of A. Then B is also a free abelian group. Furthermore, if α is the cardinality of a basis for A and β is the cardinality of a basis for B, then $\beta \le \alpha$. Any two bases of B have the same cardinality, which is called the rank of B. For more information, see Lang [4, p. 45].

4.2.5 THEOREM. The free abelian groups F_S and F_T are isomorphic if and only if rank of F_S = rank of F_T; that is, if and only if $|S| = |T|$ for the cardinals $|S|$ and $|T|$. A proof of this may be found in Fuchs [2, p. 73].

We now have the necessary machinery to prove the analogy to the Cantor-Schröder-Bernstein Theorem for free abelian groups.

4.2.6 THEOREM. Let F and G be free abelian groups. If α is a (group) isomorphism from F into G and if β is a (group) isomorphism from G into F, then there exists a (group) isomorphism ϕ from F onto G.

Proof: Let F and G be free abelian groups of rank n and m, respectively. Let α be an isomorphism from F into G. Then $F \cong \operatorname{Im}\alpha$, and by 4.2.5, rank of F = rank of $\operatorname{Im}\alpha = |n|$. By 4.2.4, $\operatorname{Im}\alpha$ is a free abelian group, and rank of $\operatorname{Im}\alpha \le |m|$. Hence, $|n| \le |m|$.

Let β be an isomrophism from G into F. Then $G \cong \operatorname{Im} \beta$, and by 4.2.5, rank of $\operatorname{Im} \beta = |m|$. By 4.2.4, rank of $\operatorname{Im} \beta \leq |n|$. Hence $|m| \leq |n|$.

Since $|n| \leq |m|$ and $|m| \leq |n|$, $|n| = |m|$ and by 4.2.5, $F \cong G$. □

Notice that this method of proof is very similar to the proof for vector spaces, as it relies on the property of having a basis. Furthermore, this method of proof will work only for free abelian groups, since by 4.2.3 they are the only abelian groups which have a basis.

SECTION 3: Divisible Groups

4.3.1 DEFINITION. Let Z^+ denote the positive integers. An abelian group D is divisible provided if $n \in Z^+$ and if $x \in D$, then there exists a $y \in D$ such that $ny = x$.

4.3.2 DEFINITION. Let p be a prime number. If n is a positive integer, we identify the group Z_{p^n} with a subgroup of the group $Z_{p^{n+1}}$, and we consider Z_{p^n} to be a subgroup of $Z_{p^{n+1}}$. Thus we have the chain

$$Z_p \subseteq Z_{p^2} \subseteq Z_{p^3} \subseteq \cdots \subseteq Z_{p^n} \subseteq Z_{p^{n+1}} \subseteq \cdots .$$

Let $Z(p^\infty) = \bigcup_{i=1}^{\infty} Z_{p^i}$; then the abelian group $Z(p^\infty)$ is called a quasicyclic group, or a group of type p^∞.

One can prove that $Z(p^\infty)$ is generated by the elements $c_1, c_2, \cdots, c_n, \cdots$, where each c_n generates the subgroup Z_{p^n}. Furthermore, $pc_1 = 0$, $pc_2 = c_1$, and in general, $pc_{n+1} = c_n$. Also one can prove that every element of $Z(p^\infty)$ is a multiple of c_n for some n, and that $Z(p^\infty)$ is a divisible group. For a further discussion, see Fuchs [2, p. 15].

4.3.3 EXAMPLE. Let $p = 2$. Then $Z_2 \subseteq Z_4 \subseteq Z_8 \subseteq \cdots \subseteq Z_{2^n} \subseteq \cdots$, and $Z(2^\infty) = \bigcup_{n=1}^{\infty} Z_{2^n}$.

The following three theorems are necessary for our analogy. Proofs of the theorems may be found in Fuchs [2, p. 98 and p. 104].

4.3.4 THEOREM. If H and K are abelian groups, if H is a subgroup of K, and if H is divisible, then H is a direct summand of K, so $K \cong H \oplus L$ for some abelian group L.

4.3.5 THEOREM. If D is a divisible group, then D is a direct sum of quasicyclic and full rational groups Q; or,

$$D \cong Z(p_1^\infty) \oplus Z(p_2^\infty) \oplus \cdots \oplus Q \oplus Q \oplus \cdots \oplus Q$$
$$= \oplus \Sigma_I \; Z(p_i^\infty) \oplus \oplus \Sigma_A \; Q.$$

Furthermore, the groups $Z(p_i^\infty)$ and the cardinal number of the set A of components of rationals is unique.

4.3.6 THEOREM. The homomorphic image of a divisible group is divisible, and a direct summand of a divisible group is divisible.

We now are ready to prove an analogy to the Cantor-Schröder-Bernstein Theorem for divisible groups.

4.3.7 THEOREM. Let G and H be two divisible groups. If G is isomorphic to a subgroup of H, and if H is isomorphic to a subgroup of G, then G is isomorphic to H.

Proof: Let G and H be two divisible groups. Let G be isomorphic to a subgroup of H, denoted Im (G). By Theorems 4.3.4 and 4.3.6, Im (G) (which is isomorphic to G) is a direct summand of H, so $H \cong G \oplus K$ for some abelian group K.

Let H be isomorphic to a subgroup of G, denoted Im (H), which is isomorphic to H. Again by Theorems 4.3.4 and 4.3.6, $G \cong H \oplus L$ for some abelian group K.

Note that L and K are divisible, since every direct summand of a divisible group is divisible by 4.3.6. Let

(*) $G = \oplus \Sigma_I Z(p_i^\infty) \oplus \oplus \Sigma_A Q$ and $H = \oplus \Sigma_J Z(q_j^\infty) \oplus \oplus \Sigma_B Q$, where I, J, A and B are unique index sets, and p_i and q_j are prime numbers for each $i \in I$ and each $j \in J$. Furthermore, the groups $\oplus \Sigma_I Z(p_i^\infty)$ and $\oplus \Sigma_J Z(q_j^\infty)$ are unique, except for order.

Now, let $L = \oplus \Sigma_{I'} Z(p_i'^\infty) \oplus \oplus \Sigma_C Q$ and $K = \oplus \Sigma_{J'} Z(q_j'^\infty) \oplus \oplus \Sigma_D Q$, for index sets I', J', C and D, and for prime numbers p'_i and q'_j for each $i \in I'$ and each $j \in J'$. Then

$G = H \oplus L = \oplus \Sigma_J Z(q_j^\infty) \oplus \oplus \Sigma_B Q \oplus \Sigma_{I'} Z(p_i'^\infty) \oplus \oplus \Sigma_C Q$. Thus by (*) and the uniqueness of the index sets (4.3.5), $|B| + |C| = |A|$, where $|A|$ denotes the cardinal number of A, and

$\{Z(q_j^\infty) \mid j \in J\} \cup \{Z(p_i'^\infty) \mid i \in I'\} = \{Z(p_i^\infty) \mid i \in I\}$. In particular, $|B| \leq |A|$ and $\{Z(q_j^\infty) \mid j \in J\} \subseteq \{Z(p_i^\infty) \mid i \in I\}$. But

$H = G \oplus K = \oplus \Sigma_I Z(p_i^\infty) \oplus \oplus \Sigma_A Q \oplus \oplus \Sigma_{J'} Z(q_j'^\infty) \oplus \oplus \Sigma_D Q$. Hence by (*) and (4.3.5), $|A| + |D| = |B|$, or $|A| \leq |B|$. Therefore $|A| = |B|$.

Furthermore,

$$\{Z(p_i^\infty) \mid i \in I\} \subseteq \{Z(q_j^\infty) \mid j \in J\}.$$

Hence,

$$\{Z(p_i^\infty) \mid i \in I\} = \{Z(q_j^\infty) \mid j \in J\}.$$

Thus $G \cong H$. □

SUMMARY

Beginning with the conventional form of the Cantor-Schröder-Bernstein Theorem in set theory, we proceeded to investigate five other mathematical settings to see if analogous theorems could be formulated in these settings. We proved that analogous theorems exist in vector spaces, in finite groups, in free abelian groups, and in divisible groups. However, counterexamples were presented to demonstrate that no theorems analogous to the Cantor-Schröder-Bernstein Theorem could be formulated for topological spaces or for groups of infinite order.

We can ask if a theorem analogous to the Cantor-Schröder-Bernstein Theorem is true in virtually any area of mathematics, so that we could never hope to answer every conjecture of this type. However, two questions which we would have liked to investigate, given sufficient time, are the following:

For what special classes of topological spaces does the analogy hold true, and for what special classes of modules over a ring does the analogy hold true?

BIBLIOGRAPHY

1. R. H. Cox, Schröder-Bernstein Theórem, American Mathematical Monthly, Vol. 75, May 1968, p. 508.

2. Lászlo Fuchs, Infinite Abelian Groups, Vol. I, Academic Press, New York and London, 1970.

3. Kenneth Hoffman and Ray Kunze, Linear Algebra, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

4. Serge Lang, Algebra, Addison-Wesley Publishing Co., Reading, Massachusetts, 1965.

5. Stephen Willard, General Topology, Addison-Wesley Publishing Co., Reading, Massachusetts, 1970.